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Adiabatic invariants of some time-dependent oscillators

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Abstract
Two types of time-dependent 1D oscillators are considered: parametrically excited, described by the Mathieu equation, and parametrically pumped with a periodic driving force. The adopted field method approach is applied and the complete sets of their linear invariants as well as the corresponding quadratic invariants are derived.

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1. Introduction

The systems whose behaviour is modelled by

$$\ddot{x} + \omega^2(t)x = 0 \tag{1}$$

play a very important role in various branches of physics (classical and quantum mechanics, for example) as well as in engineering. In classical mechanics, equation (1) models the motion of a slowly lengthening pendulum or an oscillator whose position is defined by the generalized coordinate x, while its frequency ω varies in time. In quantum mechanics, equation (1) matches a one-dimensional stationary Schrödinger equation in which the probability amplitude ψ is replaced by the coordinate x, while the coordinate appearing therein is replaced by time t and ω^2 is equivalent to the subtraction of the energy and the potential [1]. In engineering, particularly in mechanical and electrical engineering, it describes a variety of systems whose parameters change slowly with respect to time.

A large number of varied approaches to obtaining adiabatic invariants of time-dependent oscillators, the review of which is given in [2], point out the theoretical and practical significance of the knowledge of these quantities which remain approximately conserved when the system parameters are slowly varying or, more precisely, whose time derivative contains only terms higher than some particular order.

The present study is aimed at obtaining adiabatic invariants of some time-dependent 1D oscillators: parametrically excited, described by the Mathieu equation, and parametrically pumped with a periodic driving force. This study is a natural continuation of recent research on the possibility of finding these quantities [3] by applying the field method, the basics of which will be given in the next section.

2. On the field method

A starting point of the field method algorithm [4–8] for studying motion of any mechanical system, say the time-dependent oscillator (1), is representing its mathematical model in the form of the first-order differential equations:

$$\dot{x} = p \qquad \dot{p} = -\omega^2(t)x. \tag{2}$$

Then, the field is chosen from the set of the state variables, i.e. it can be a generalized coordinate x or momentum p. In such a way both state variables are treated equivalently and two approaches can be taken. Namely, it can be assumed that the momentum can be expressed in terms of time and the generalized coordinate $p = \Phi(t, x)$, which represents the so-called field-momentum approach. On the other hand, following the field-coordinate approach, the generalized coordinate x can be nominated to form a field depending on time and the momentum $x = U(t, p)$. The first approach seems to be more reasonable and comparable to that used, for example, in the Hamilton–Jacobi theory. The second one is motivated by the fact that in analytical mechanics both coordinates and momentums play an equal role in describing the motion [4]. Since the field-momentum approach has been more frequently applied to many different and disparate problems of mechanics, in this study the field-coordinate approach will be used in order to show that it can also be beneficial.

Partial differentiation of the expression $x = U(t, p)$ in combination with (2) yields the so-called basic field equation:

$$\frac{\partial U}{\partial t} - \frac{\partial U}{\partial p}\omega^2(t)U - p = 0. \tag{3}$$

For the case when $\omega^2 = \text{const}$, the corresponding basic field equation is a quasi-linear partial differential equation, which, according to invariant embedding theory [9], may be integrated by assuming a general solution as an affine transformation:

$$U = f_1(t)p + f_2(t). \tag{4}$$

Besides, it has been shown [5, 7] that if one assumes an incomplete solution of such an equation in the form in which one arbitrary constant A appears,

$$U = Ap + f(t), \tag{5}$$

an invariant (conservation law) of the system can be obtained.

We will demonstrate that an approximate incomplete solution of the basic field equation (3) leads to a complete set of adiabatic invariants of the system (1). In this way, complete integrability is established and an approximate solution for motion is specified. In order to accomplish the approximate incomplete solution, the multiple variable expansion procedure will be combined with the field-coordinate approach.

3. Parametrically excited system

Consider an oscillator whose frequency changes according to

$$\omega^2(t) = \delta + 2\varepsilon\cos 2t, \tag{6}$$

where δ is the frequency parameter and ε is a small constant parameter ($\varepsilon \ll 1$). The corresponding differential equation of motion (the Mathieu equation) is the archetype for a linear system with an internal periodic excitation. The boundness (stability) of its solutions has been studied extensively, including obtaining the so-called transitional curves in the $\delta\varepsilon$-plane [10, 11]. These curves separate the regions in which motion is unbounded (unstable) from the region in which all solutions are bounded (stable). Besides, the asymptotic periodic solutions along transitional curves have been of much of interest. Among many procedures for deriving them, the most frequently used are the perturbational methods: the method of strained parameters and the method of multiple scales [10]. In the following study, the adiabatic invariants which hold right on the transitional curves will be determined by combining the field method technique with the perturbational procedure.

The equivalent to the Mathieu equation in the field method approach is the basic field equation:

$$\frac{\partial U}{\partial t} - \frac{\partial U}{\partial p}(\delta + 2\varepsilon \cos 2t)U - p = 0. \tag{7}$$

In order to find an approximate solution of this partial differential equation, the following power expansions in ε are used for time, the field, the other state variable as well as for the transitional curves:

$$T = t, \qquad \tau = \varepsilon t, \qquad \tilde{\tau} = \varepsilon^2 t, \tag{8}$$

$$U(t, p, \varepsilon) = U_0(T, \tau, p_0) + \varepsilon U_1(T, \tau, p_1) + \varepsilon^2 U_2(T, \tau, p_2), \tag{9}$$

$$p(t, \varepsilon) = p_0(T, \tau) + \varepsilon p_1(T, \tau) + \varepsilon^2 p_2(T, \tau), \tag{10}$$

$$\delta = \delta_0 + \varepsilon\delta_1 + \varepsilon^2\delta_2. \tag{11}$$

Besides, it assumes that the dependence of the field on the corresponding variable is not affected by the step of approximation [6], i.e.,

$$\frac{\partial U}{\partial p} = \frac{\partial U_0}{\partial p_0} = \frac{\partial U_1}{\partial p_1} = \frac{\partial U_2}{\partial p_2}. \tag{12}$$

Substituting (8)–(12) into (7) and equating coefficient of like powers of ε, one obtains

$$\frac{\partial U_0}{\partial T} - \frac{\partial U_0}{\partial p_0}\delta_0 U_0 - p_0 = 0, \tag{13}$$

$$\frac{\partial U_1}{\partial T} - \frac{\partial U_1}{\partial p_1}\delta_0 U_1 - p_1 = -\frac{\partial U_0}{\partial \tau} + \frac{\partial U_0}{\partial p_0}\delta_1 U_0^* + \frac{\partial U_0}{\partial p_0}U_0^*(\mathrm{e}^{2\mathrm{i}T} + \mathrm{e}^{-2\mathrm{i}T}), \tag{14}$$

$$\frac{\partial U_2}{\partial T} - \frac{\partial U_2}{\partial p_2}\delta_0 U_2 - p_2 = -\frac{\partial U_1}{\partial \tau} + \frac{\partial U_1}{\partial p_1}\delta_1 U_1^* + \frac{\partial U_0}{\partial p_0}\delta_2 U_0^* + \frac{\partial U_1}{\partial p_1}U_1^*(\mathrm{e}^{2\mathrm{i}T} + \mathrm{e}^{-2\mathrm{i}T}), \tag{15}$$

where i is an imaginary unit. In this coupled system of partial differential equations, U_0–U_2 represent the incomplete solutions, while U_0^* and U_1^* are the so-called solutions along trajectories. Unlike the incomplete solutions, which depend on time and the other state variable (9), the solutions along trajectories should be expressed as functions of time only.

It is noticeable that the form of (13) as well as the left-hand sides of (14) and (15) corresponds to (3) with ω^2 = const, whose incomplete solution can be assumed as (5). Following this analogy, but keeping in mind the existence of time scale (8), the trial incomplete solutions for the field components are assumed as [3]

$$U_0 = Ap_0 + C_0(\tau)\,\mathrm{e}^{A\delta_0 T}, \tag{16}$$

$$U_j = Ap_j + C_j(T, \tau)\,\mathrm{e}^{A\delta_0 T}, \qquad j = 1, 2, \tag{17}$$

where the constant A has two values:

$$A' = \frac{\mathrm{i}}{\sqrt{\delta_0}}, \qquad A'' = -\frac{\mathrm{i}}{\sqrt{\delta_0}}. \tag{18}$$

Note that the technique used in this procedure regarding supposing solutions in the forms (8)–(11) and (16), (17) is based on the fundamental assumption generally accepted in all asymptotic methods that the successive approximations are developed by a recursive procedure for a fixed number of terms and for $\varepsilon \to 0$, so that these expansions represent reliable solutions on some finite interval of time and reasonably correct solutions for the long interval of time.

Two values of the constant A (18) imply that all unknown functions C_α ($\alpha = 0, 1, 2$) in (16), (17) can have two forms:

$$C'_\alpha = C_\alpha(A'), \qquad C''_\alpha = C_\alpha(A''). \tag{19}$$

If the initial conditions are prescribed as $x(t = 0) = a$ and $p(t = 0) = b$, expressions (16), (17) imply

$$C'_0(0) = a - \frac{\mathrm{i}b}{\sqrt{\delta_0}}, \qquad C''_0(0) = a + \frac{\mathrm{i}b}{\sqrt{\delta_0}}, \tag{20}$$

$$C'_j(0, 0) = C''_j(0, 0) = 0. \tag{21}$$

Then, the incomplete solutions will be transformed into the solutions along trajectories U_0^* and U_1^*. Namely, taking into account (18) and (19), equations (16) and (17) give

$$U_0^* = \frac{C'_0(\tau)\,\mathrm{e}^{\mathrm{i}\sqrt{\delta_0}T} + C''_0(\tau)\,\mathrm{e}^{-\mathrm{i}\sqrt{\delta_0}T}}{2}, \tag{22}$$

$$U_j^* = \frac{C'_j(T,\tau)\,\mathrm{e}^{\mathrm{i}\sqrt{\delta_0}T} + C''_j(T,\tau)\,\mathrm{e}^{-\mathrm{i}\sqrt{\delta_0}T}}{2}. \tag{23}$$

Using the incomplete solutions (16), (17) and the solutions along trajectories (22), (23), equations (14), (15) transform into

$$\begin{aligned}\frac{\mathrm{d}C_1}{\mathrm{d}T} = {} & -\frac{\mathrm{d}C_0}{\mathrm{d}\tau} + A\delta_1\,\mathrm{e}^{-A\delta_0 T}\,\frac{C'_0\,\mathrm{e}^{\mathrm{i}\sqrt{\delta_0}T} + C''_0\,\mathrm{e}^{-\mathrm{i}\sqrt{\delta_0}T}}{2}\\ & + A\,\mathrm{e}^{-A\delta_0 T}\,\frac{\left(C'_0\,\mathrm{e}^{\mathrm{i}\sqrt{\delta_0}T} + C''_0\,\mathrm{e}^{-\mathrm{i}\sqrt{\delta_0}T}\right)(\mathrm{e}^{2\mathrm{i}T} + \mathrm{e}^{-2\mathrm{i}T})}{2},\end{aligned} \tag{24}$$

$$\begin{aligned}\frac{\mathrm{d}C_2}{\mathrm{d}T} = {} & -\frac{\mathrm{d}C_1}{\mathrm{d}\tau} + A\delta_1\,\mathrm{e}^{-A\delta_0 T}\,\frac{C'_1\,\mathrm{e}^{\mathrm{i}\sqrt{\delta_0}T} + C''_1\,\mathrm{e}^{-\mathrm{i}\sqrt{\delta_0}T}}{2} + A\delta_2\,\mathrm{e}^{-A\delta_0 T}\,\frac{C'_0\,\mathrm{e}^{\mathrm{i}\sqrt{\delta_0}T} + C''_1\,\mathrm{e}^{-\mathrm{i}\sqrt{\delta_0}T}}{2}\\ & + A\,\mathrm{e}^{-A\delta_0 T}\,\frac{\left(C'_1\,\mathrm{e}^{\mathrm{i}\sqrt{\delta_0}T} + C''_1\,\mathrm{e}^{-\mathrm{i}\sqrt{\delta_0}T}\right)(\mathrm{e}^{2\mathrm{i}T} + \mathrm{e}^{-2\mathrm{i}T})}{2}.\end{aligned} \tag{25}$$

Next, the requirement of no appearance of the secular terms will be used to find C'_0, C''_0 from (24) and C'_1, C''_1 from (25). On analysis of these two equations, it is clear that the secular terms depend primarily on the value of δ_0. The obvious critical value is $\delta_0 = 1$, which corresponds to the periodic solutions of period 2π (22), (23). The case $\delta_0 = 4$ corresponds to the periodic solutions with period π. This implies that the further analysis should be carried out separately starting from these particular values of δ_0. But before that, let derive the general forms of the adiabatic invariants as the functions of C'_0, C''_0, C'_1 and C''_1.

3.1. On the forms of the adiabatic invariants

The field (9) limited to the second component (17) is

$$x \equiv U = Ap_0 + C_0(\tau)\,\mathrm{e}^{A\delta_0 T} + \varepsilon[Ap_1 + C_1(T,\tau)\,\mathrm{e}^{A\delta_0 T}]. \tag{26}$$

Including two possible values of A, C_0 and C_1, as well as recognizing the expansion (10) for p, two independent complex adiabatic invariants can be obtained:

$$\left[x - \frac{ip}{\sqrt{\delta_0}}\right]\mathrm{e}^{-\mathrm{i}\sqrt{\delta_0}T} - C_0'(\tau) - \varepsilon C_1'(T,\tau) = 0, \tag{27}$$

$$\left[x + \frac{ip}{\sqrt{\delta_0}}\right]\mathrm{e}^{\mathrm{i}\sqrt{\delta_0}T} - C_0''(\tau) - \varepsilon C_1''(T,\tau) = 0. \tag{28}$$

Combining the invariants (27) and (28), the solution for motion can be found. Besides, their product can yield the following quadratic energy-like adiabatic invariant:

$$I \equiv x^2 + \frac{p^2}{\delta_0} - \varepsilon[C_0'(\tau)C_1''(T,\tau) + C_0''(\tau)C_1'(T,\tau)] \\ + \varepsilon^2 C_1'(T,\tau)C_1''(T,\tau) - C_0'(\tau)C_0''(\tau) = 0, \tag{29}$$

or

$$\tilde{I} \equiv x^2 + \frac{p^2}{\delta_0} - \varepsilon x\big[C_1'\,\mathrm{e}^{\mathrm{i}\sqrt{\delta_0}T} + C_1''\,\mathrm{e}^{-\mathrm{i}\sqrt{\delta_0}T}\big] \\ - \varepsilon\frac{ip}{\sqrt{\delta_0}}\big[C_1'\,\mathrm{e}^{\mathrm{i}\sqrt{\delta_0}T} - C_1''\,\mathrm{e}^{-\mathrm{i}\sqrt{\delta_0}T}\big] + \varepsilon^2 C_1'C_1'' - C_0'C_0'' = 0. \tag{30}$$

In the form (29), the coefficients next to the small parameter are expressed in terms of time, while in the second form (30) these coefficients contain time and the state variables x and p.

Now, the expressions for the linear invariants and the quadratic one will be specified for some values of the parameter δ_0.

3.1.1. Case 1: $\delta_0 = 1$. In order to prevent the appearance of the secular terms in (24) when $\delta_0 = 1$, the following condition must be satisfied:

$$\frac{\mathrm{d}C_0'}{\mathrm{d}\tau} - \mathrm{i}\delta_1\frac{C_0'}{2} - \mathrm{i}\frac{C_0''}{2} = 0, \qquad \frac{\mathrm{d}C_0''}{\mathrm{d}\tau} + \mathrm{i}\delta_1\frac{C_0''}{2} + \mathrm{i}\frac{C_0'}{2} = 0. \tag{31}$$

Using $C_0' = c'\,\mathrm{e}^{r\tau}$, $C_0'' = c''\,\mathrm{e}^{r\tau}$, the system (31) gives $r^2 = \frac{1-\delta_1^2}{4}$. For the case when $|\delta_1| < 1$, two real roots exist and motion is unbounded. If $|\delta_1| > 1$, motion is bounded, having the form of modulated oscillations. For the case $|\delta_1| = 1$, periodic solutions appear along the transitional curves. Then, the functions C_0' and C_0'' are pure constants, equal to their initial values (20). When $\delta_1 = 1$, it must be $a = 0$, while for $\delta_1 = -1$ the condition $b = 0$ holds. Solving (24) for the former condition, one has

$$C_1' = \frac{C_0'}{4}\,\mathrm{e}^{2\mathrm{i}T} - \frac{C_0''}{8}\,\mathrm{e}^{-4\mathrm{i}T} + R'(\tau), \qquad C_1'' = -\frac{C_0'}{8}\,\mathrm{e}^{4\mathrm{i}T} + \frac{C_0''}{4}\,\mathrm{e}^{-2\mathrm{i}T} + R''(\tau), \tag{32}$$

where R' and R'' are unknown functions. Substituting (32) into (25) and equating secular terms with zero, one gets

$$\frac{\mathrm{d}R'}{\mathrm{d}\tau} - \mathrm{i}\frac{R'}{2} - \mathrm{i}\frac{R''}{2} - \mathrm{i}\frac{C_0'}{2}\left(\delta_2 + \frac{1}{8}\right) = 0, \\ \frac{\mathrm{d}R''}{\mathrm{d}\tau} + \mathrm{i}\frac{R'}{2} + \mathrm{i}\frac{R''}{2} + \mathrm{i}\frac{C_0''}{2}\left(\delta_2 + \frac{1}{8}\right) = 0. \tag{33}$$

This system, together with (21), gives $\delta_2 = -\frac{1}{8}$, $R' = R'(0) = \frac{3ib}{8}$, $R'' = R''(0) = -\frac{3ib}{8}$.

So, along the transition curve

$$\delta = 1 + \varepsilon - \frac{1}{8}\varepsilon^2, \tag{34}$$

the following linear adiabatic invariants exist:

$$[x - \mathrm{i}p]\,\mathrm{e}^{-\mathrm{i}T} + \varepsilon\left[\frac{\mathrm{i}b}{4}\,\mathrm{e}^{2\mathrm{i}T} + \frac{\mathrm{i}b}{8}\,\mathrm{e}^{-4\mathrm{i}T}\right] = -\mathrm{i}b + \varepsilon\frac{3\mathrm{i}b}{8}, \tag{35}$$

$$[x + \mathrm{i}p]\,\mathrm{e}^{\mathrm{i}T} - \varepsilon\left[\frac{\mathrm{i}b}{4}\,\mathrm{e}^{-2\mathrm{i}T} + \frac{\mathrm{i}b}{8}\,\mathrm{e}^{4\mathrm{i}T}\right] = \mathrm{i}b - \varepsilon\frac{3\mathrm{i}b}{8}. \tag{36}$$

Separating the real and imaginary parts, they can be expressed as

$$J_{11} \equiv x\cos T - p\sin T - \varepsilon\left[\frac{b}{4}\sin 2T - \frac{b}{8}\sin 4T\right] = 0, \tag{37}$$

$$J_{12} \equiv x\sin T + p\cos T - \varepsilon\left[\frac{b}{4}\cos 2T + \frac{b}{8}\cos 4T\right] = b - \varepsilon\frac{3b}{8}. \tag{38}$$

The corresponding quadratic invariant is

$$J_{13} \equiv x^2 + p^2 - \varepsilon\left[\frac{b^2}{2}\cos 2T + \frac{b^2}{4}\cos 4T\right] - \varepsilon^2\left[-\frac{3b^2}{16}\cos 2T - \frac{3b^2}{32}\cos 4T + \frac{b^2}{16}\cos 6T\right]$$
$$= b^2 - \varepsilon\frac{3b^2}{4} + \varepsilon^2\frac{7b^2}{32}. \tag{39}$$

Having found a complete set of adiabatic invariants (35), (36) or (37), (38), the approximate solution can also be derived.

If $\delta_1 = -1$ and $b = 0$, we have $C_0' = C_0'(0) = a$, $R' = R'' = -\frac{a}{8}$. Thus, the transition curve is given by

$$\delta = 1 - \varepsilon - \tfrac{1}{8}\varepsilon^2, \tag{40}$$

and the adiabatic invariants which hold along this curve are

$$[x - \mathrm{i}p]\,\mathrm{e}^{-\mathrm{i}T} + \varepsilon\left[\frac{a}{4}\,\mathrm{e}^{2\mathrm{i}T} - \frac{a}{8}\,\mathrm{e}^{-4\mathrm{i}T}\right] = a - \varepsilon\frac{a}{8}, \tag{41}$$

$$[x + \mathrm{i}p]\,\mathrm{e}^{\mathrm{i}T} - \varepsilon\left[\frac{a}{4}\,\mathrm{e}^{-2\mathrm{i}T} - \frac{a}{8}\,\mathrm{e}^{4\mathrm{i}T}\right] = a - \varepsilon\frac{a}{8}, \tag{42}$$

or

$$I_{11} \equiv x\cos T - p\sin T - \varepsilon\left[\frac{a}{4}\cos 2T - \frac{a}{8}\cos 4T\right] = a - \varepsilon\frac{a}{8}, \tag{43}$$

$$I_{12} \equiv x\sin T + p\cos T + \varepsilon\left[\frac{a}{4}\sin 2T + \frac{a}{8}\sin 4T\right] = 0, \tag{44}$$

$$I_{13} \equiv x^2 + p^2 - \varepsilon\left[\frac{a^2}{2}\cos 2T - \frac{a^2}{4}\cos 4T\right] - \varepsilon^2\left[-\frac{a^2}{16}\cos 2T + \frac{a^2}{32}\cos 4T - \frac{a^2}{16}\cos 6T\right]$$
$$= a^2 - \varepsilon\frac{a^2}{4} + \varepsilon^2\frac{3a^2}{32}. \tag{45}$$

3.1.2. Case 2: $\delta_0 = 4$. After substituting the value $\delta_0 = 4$ into (24) and using (18), one concludes that the following secular terms must vanish:

$$\frac{dC_0'}{d\tau} - i\delta_1 \frac{C_0'}{4} = 0, \qquad \frac{dC_0''}{d\tau} + i\delta_1 \frac{C_0''}{4} = 0. \tag{46}$$

It follows that $\delta_1 = 0$ and $C_0' = C_0'(0)$, $C_0'' = C_0''(0)$. Integrating what remains in (24) leads to

$$\begin{aligned} C_1' &= \frac{C_0'}{8} e^{2iT} - \frac{C_0'}{8} e^{-2iT} - \frac{C_0''}{8} e^{-2iT} - \frac{C_0''}{24} e^{-6iT} + R'(\tau), \\ C_1'' &= -\frac{C_0'}{8} e^{2iT} - \frac{C_0''}{8} e^{2iT} + \frac{C_0''}{8} e^{-2iT} - \frac{C_0'}{24} e^{6iT} + R''(\tau). \end{aligned} \tag{47}$$

Putting these expressions into (25) and eliminating the secular terms, we obtain

$$\frac{dR'}{d\tau} - i\delta_2 \frac{C_0'}{4} + \frac{i}{4}\left(\frac{C_0'}{6} + \frac{C_0''}{4}\right) = 0, \qquad \frac{dR''}{d\tau} + i\delta_2 \frac{C_0''}{4} - \frac{i}{4}\left(\frac{C_0'}{4} + \frac{C_0''}{6}\right) = 0. \tag{48}$$

For $a = 0$, we have $C_0' = -C_0'' = -\frac{ib}{2}$ and $\delta_2 = -\frac{1}{12}$, $R' = -R'' = \frac{ib}{12}$. However, when $b = 0$, the solutions are $C_0' = C_0'' = a$, $\delta_2 = \frac{5}{12}$, $R' = R'' = \frac{a}{6}$.

With the above given quantities, the transition curve is

$$\delta = 4 - \tfrac{1}{12}\varepsilon^2, \tag{49}$$

while its corresponding adiabatic invariants are

$$\left[x - \frac{ip}{2}\right] e^{-2iT} + \varepsilon \left[\frac{ib}{16} e^{2iT} + \frac{ib}{48} e^{-6iT}\right] = -\frac{ib}{2} + \varepsilon \frac{ib}{12}, \tag{50}$$

$$\left[x + \frac{ip}{2}\right] e^{2iT} - \varepsilon \left[\frac{ib}{16} e^{-2iT} + \frac{ib}{48} e^{6iT}\right] = \frac{ib}{2} - \varepsilon \frac{ib}{12}. \tag{51}$$

These complex invariants yield the real ones:

$$I_{41} \equiv x \cos 2T - \frac{p}{2} \sin 2T + \varepsilon \left[-\frac{b}{16} \sin 2T + \frac{b}{48} \sin 6T\right] = 0, \tag{52}$$

$$I_{42} \equiv x \sin 2T + \frac{p}{2} \cos 2T - \varepsilon \left[\frac{b}{16} \cos 2T + \frac{b}{48} \cos 4T\right] = \frac{b}{2} - \varepsilon \frac{b}{12}, \tag{53}$$

while the quadratic one is

$$\begin{aligned} I_{43} \equiv x^2 + \frac{p^2}{4} - \varepsilon \left[\frac{b^2}{16} \cos 2T + \frac{b^2}{48} \cos 6T\right] - \varepsilon^2 \Bigg[-\frac{b^2}{96} \cos 2T - \frac{b^2}{288} \cos 6T \\ + \frac{b^2}{384} \cos 8T\Bigg] = \frac{b^2}{4} - \varepsilon \frac{b^2}{12} + \varepsilon^2 \frac{13b^2}{1152}. \end{aligned} \tag{54}$$

Another transition curve emanating form the same origin is given by

$$\delta = 4 + \frac{5}{12}\varepsilon^2, \tag{55}$$

and the adiabatic invariants along this curve are

$$\left[x - \frac{ip}{2}\right] e^{-2iT} - \varepsilon \left[\frac{a}{8} e^{2iT} - \frac{a}{4} e^{-2iT} - \frac{a}{24} e^{-6iT}\right] = a + \varepsilon \frac{a}{6}, \tag{56}$$

$$\left[x + \frac{ip}{2}\right] e^{2iT} - \varepsilon \left[\frac{a}{8} e^{-2iT} - \frac{a}{4} e^{2iT} - \frac{a}{24} e^{6iT}\right] = a + \varepsilon \frac{a}{6}, \tag{57}$$

i.e.,

$$J_{41} \equiv x \cos 2T - \frac{p}{2} \sin 2T + \varepsilon \left[\frac{a}{8} \cos 2T + \frac{a}{24} \cos 6T\right] = a + \varepsilon \frac{a}{6}, \tag{58}$$

$$J_{42} \equiv x \sin 2T + \frac{p}{2}\cos 2T + \varepsilon\left[\frac{3a}{8}\sin 2T + \frac{a}{24}\sin 6T\right] = 0. \tag{59}$$

The quadratic invariant is

$$J_{43} \equiv x^2 + \frac{p^2}{4} + \varepsilon\left[\frac{a^2}{4}\cos 2T + \frac{a^2}{12}\cos 6T\right] - \varepsilon^2\left[-\frac{a^2}{24}\cos 2T - \frac{a^2}{24}\cos 4T - \frac{a^2}{72}\cos 6T - \frac{a^2}{96}\cos 8T\right] = a^2 + \varepsilon\frac{a^2}{3} + \varepsilon^2\frac{31a^2}{288}. \tag{60}$$

3.1.3. Case 3: $\delta_0 = k^2$, $k \geqslant 3$. When the parameter k has odd values the periodic solutions of period 2π exist, while the even values of k result in the periodic solutions with period π. Following the algorithm proposed above, the expression for the transition curves can be obtained:

$$\delta = k^2 + \frac{1}{2(k^2-1)}\varepsilon^2, \tag{61}$$

as well as the linear adiabatic invariants:

$$\left[x - \frac{\mathrm{i}p}{k}\right]\mathrm{e}^{-\mathrm{i}kT} = C_0' + \frac{\varepsilon}{k}\left[\frac{C_0'}{4}\mathrm{e}^{2\mathrm{i}T} - \frac{C_0'}{4}\mathrm{e}^{-2\mathrm{i}T} - \frac{C_0''}{4(k-1)}\mathrm{e}^{-2\mathrm{i}(k-1)T} - \frac{C_0''}{4(k+1)}\mathrm{e}^{-2\mathrm{i}(k+1)T} + \frac{C_0''k}{2(k^2-1)}\right], \tag{62}$$

$$\left[x + \frac{\mathrm{i}p}{k}\right]\mathrm{e}^{\mathrm{i}kT} = C_0'' - \frac{\varepsilon}{k}\left[\frac{C_0''}{4}\mathrm{e}^{2\mathrm{i}T} - \frac{C_0''}{4}\mathrm{e}^{-2\mathrm{i}T} + \frac{C_0'}{4(k-1)}\mathrm{e}^{2\mathrm{i}(k-1)T} + \frac{C_0'}{4(k+1)}\mathrm{e}^{2\mathrm{i}(k+1)T} - \frac{C_0'k}{2(k^2-1)}\right], \tag{63}$$

where $C_0' = a - \frac{ib}{k}$, $C_0'' = a + \frac{ib}{k}$.

More accurate results for the adiabatic invariants could be obtained if the calculation of the function C_2' (25), i.e. C_2' and C_2'', has been completed. So, integrating (25) with respect to T for the known forms of C_0', C_0'', C_1', C_1'' and using the condition (21), these functions could be specified. As a consequence, the terms $-\varepsilon^2 C_2'$ and $-\varepsilon^2 C_2''$ would, respectively, appear on the left-hand sides of the expressions for the adiabatic invariants (27) and (28).

3.2. Numerical results

The expressions for the transition curves (34), (40), (49), (55) and (61) are in complete agreement with the forms which can be found in the literature (for instance, in [10, 11]). However, as far as the author is aware, the expressions for the adiabatic invariants which hold along these curves have not been known previously. In order to confirm that they are indeed adiabatic invariants, i.e. that they are conserved reasonably well in the asymptotic evolution, some numerical simulations will be provided.

Using the analytical expressions for the transition curves (34), (40), (49), (55), the $\delta\varepsilon$-plane restricted to $\delta_0 = 1$ and $\delta_0 = 4$ is plotted in figure 1. Numerical simulations have been carried out for four particular points A, B, C and D located on the transition curves and the horizontal line $\varepsilon = 0.1$. (Note that the point C is on the left-hand side curve emanating from $\delta_0 = 4$, while D lies on the right-hand side curve. The initial conditions are either $a = 1$, $b = 0$ (for points A and D) or $a = 0$, $b = 1$ (for points B and C).)

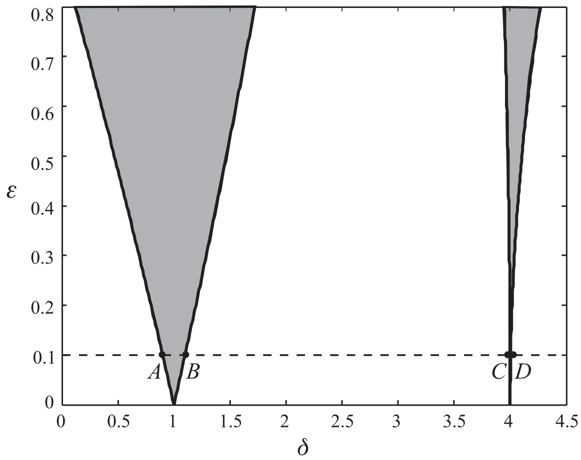

Figure 1. Transition curves for the Mathieu equation in the $\delta\varepsilon$-plane (shaded regions are the regions of unstable motion).

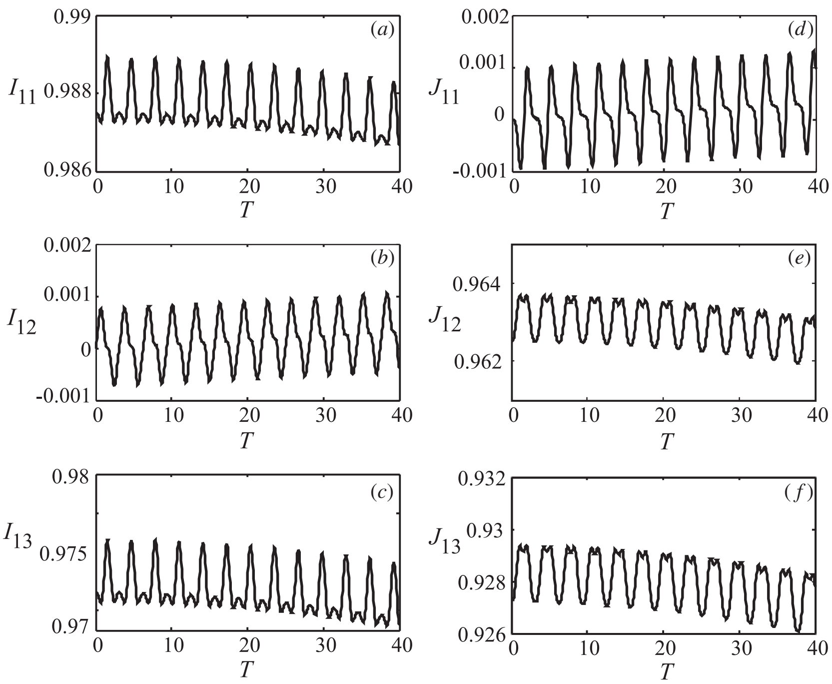

Figure 2. Adiabatic invariants: (*a*) I_{11} for point *A*, (*b*) I_{12} for point *A*, (*c*) I_{13} for point *A*, (*d*) J_{11} for point *B*, (*e*) J_{12} for point *B* and (*f*) J_{13} for point *B*.

In figures 2(a)–(c), the adiabatic invariants I_{11}(43), I_{12} (44) and I_{13} (45) corresponding to point A are shown. Figures 2(d)–(f) illustrate the adiabatic invariants J_{11} (37), J_{12} (38) and J_{13} (39) for the point B. It can be seen that all adiabatic invariants for these two points have a tendency of small change over time.

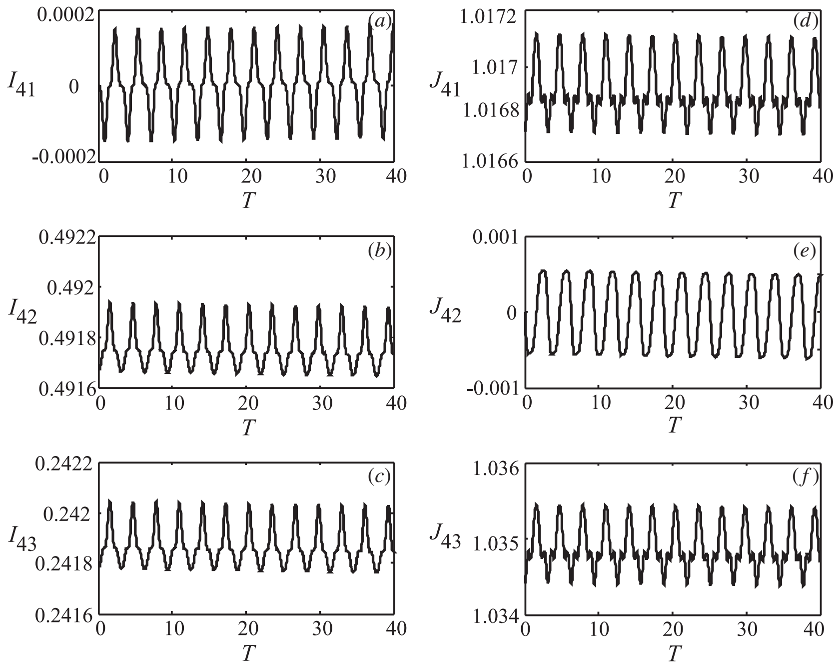

Figure 3. Adiabatic invariants: (*a*) I_{41} for point C, (*b*) I_{42} for point C, (*c*) I_{43} for point C, (*d*) J_{41} for point D, (*e*) J_{42} for point D and (*f*) J_{43} for point D.

The adiabatic invariants for the points C (52)–(54) and D (58)–(60) have the same property, as it is, respectively, shown in figures 3(a)–(c) and(d)–(f).

4. Parametrically pumped system with a periodic driving force

The system described by the Hamiltonian

$$H = \frac{1}{2}\frac{p^2}{m(t)} + \frac{1}{2}m(t)\omega^2 x^2, \tag{64}$$

with a pulsating mass

$$m(t) = m_0\,\mathrm{e}^{2\varepsilon\sin\upsilon t}, \qquad \varepsilon \ll 1, \tag{65}$$

where ω, m_0 and υ are constants, is an example of a parametrically pumped oscillator. Such a mass pumping function also describes, for instance, the problem in a quantum optics of a cavity field pumped by an atomic reservoir or an electric circuit with weakly pumped capacitance [12, 13]. An energy-like invariant for such systems is given in [14, 15]. In the succeeding study, we will propose the procedure for deriving invariants of the driven pumped system described by the Kanai–Caldirola Hamiltonian:

$$H = \frac{1}{2}\frac{p^2}{m(t)} + \frac{1}{2}m(t)\omega^2 x^2 - m(t)F(t)x, \tag{66}$$

where $F(t) = F_0 \sin\Omega t$ is the periodic driving force.

Passing to a new coordinate:

$$x = X\,\mathrm{e}^{-\varepsilon \sin \upsilon t}, \tag{67}$$

the corresponding equation of motion to first order in ε can be presented in the form

$$\begin{aligned} \dot{X} &= P, \\ \dot{P} &= -(\omega^2 + \varepsilon\upsilon^2 \sin \upsilon t)X + F_0(1 + \varepsilon \sin \upsilon t) \sin \Omega t. \end{aligned} \tag{68}$$

Obviously, the coefficient next to the new coordinate X is now time dependent and the system (68) represents the non-homogeneous equation of motion of time-dependent 1D harmonic oscillator (1). Choosing the field $X = U(t, P)$, the basic field equation reads

$$\frac{\partial U}{\partial t} + \frac{\partial U}{\partial P}[-(\omega^2 + \varepsilon\upsilon^2 \sin \upsilon t)U + F_0(1 + \varepsilon \sin \upsilon t) \sin \Omega t] - P = 0. \tag{69}$$

Representing time t, the field U and the variable P analogously to the expansions (8)–(10), one obtains

$$\frac{\partial U_0}{\partial T} - \frac{\partial U_0}{\partial P_0}[\omega^2 U_0 - F_0 \sin \Omega T] - P_0 = 0, \tag{70}$$

$$\begin{aligned} \frac{\partial U_1}{\partial T} - \frac{\partial U_1}{\partial P_1}\omega^2 U_1 - P_1 = &- \frac{\partial U_0}{\partial \tau} + \frac{\partial U_0}{\partial P_0}\upsilon^2 U_0^* \frac{\mathrm{e}^{\mathrm{i}\upsilon T} - \mathrm{e}^{-\mathrm{i}\upsilon T}}{2i} \\ &+ \frac{\partial U_0}{\partial P_0} F_0 \frac{\mathrm{e}^{\mathrm{i}\Omega T} - \mathrm{e}^{-\mathrm{i}\Omega T}}{2} \frac{\mathrm{e}^{\mathrm{i}\upsilon T} - \mathrm{e}^{-\mathrm{i}\upsilon T}}{2}, \end{aligned} \tag{71}$$

$$\frac{\partial U_2}{\partial T} - \frac{\partial U_2}{\partial P_2}\omega^2 U_2 - P_2 = - \frac{\partial U_1}{\partial \tau} + \frac{\partial U_1}{\partial P_1}\upsilon^2 U_1^* \frac{\mathrm{e}^{\mathrm{i}\upsilon T} - \mathrm{e}^{-\mathrm{i}\upsilon T}}{2i}. \tag{72}$$

Knowing that the parametric resonance $\upsilon = 2\omega$ is illusory from a physical point of view [14], this system will be studied from the standpoint of the influence of the driving force. Two cases will be analysed separately: the non-resonant case and the case when the forced resonance occurs.

4.1. Non-resonant force

The adiabatic invariants will be sought firstly for the case when $\Omega \neq \omega$.

The partial differential equation for the first component U_0 (70) is equivalent to the basic field equation of the forced linear oscillator. Namely, if $\varepsilon = 0$, the system (68) models the forced linear oscillator and the corresponding basic field equation (69) coincides with (70).

Taking the solution of (70) as

$$U_0 = BP_0 + f(T, \tau), \tag{73}$$

where B is a constant and $f(T, \tau)$ is an unknown time-depending function, substituting it into (70), separating the terms containing P_0 and the free terms with zero, one obtains that the constant B can have two values:

$$B' = \frac{\mathrm{i}}{\omega}, \qquad B'' = -\frac{\mathrm{i}}{\omega}, \tag{74}$$

while

$$f(T, \tau) = C_0(\tau)\,\mathrm{e}^{B\omega^2 T} + \frac{BF_0}{2}\left[\frac{\mathrm{e}^{B\omega\Omega T}}{\Omega - \omega} + \frac{\mathrm{e}^{-B\omega\Omega T}}{\Omega + \omega}\right], \tag{75}$$

where $C_0(\tau)$ is a function to be found.

From (73)–(75), the solutions along trajectories U_0^* are derived:

$$U_0^* = \frac{C_0'(\tau)\,\mathrm{e}^{\mathrm{i}\omega T} + C_0''(\tau)\,\mathrm{e}^{-\mathrm{i}\omega T}}{2} - \frac{F_0}{\Omega^2-\omega^2}\frac{\mathrm{e}^{\mathrm{i}\Omega T}-\mathrm{e}^{-\mathrm{i}\Omega T}}{2\mathrm{i}}. \tag{76}$$

Further, on the basis of the analogy between (14), (15) and (71)–(72), the forms of the solutions for the second and third components of the field are assumed as

$$U_j = BP_j + C_j(T,\tau)\,\mathrm{e}^{B\omega^2 T}, \qquad j = 1, 2, \tag{77}$$

with $C_j(T,\tau)$ being unknown, while the solutions along trajectory U_j^* are equal to (23), for $\omega^2 \equiv \delta_0$.

The elimination of the secular terms in equation (71) leads to the conclusion that C_0' and C_0'' are constant. If the initial conditions are defined so that $X(t=0)=a$ and $P(t=0)=b$, the solution along trajectory (76) gives

$$C_0' = a + \frac{b}{\omega\mathrm{i}} + \frac{F_0\Omega}{\omega(\Omega^2-\omega^2)\mathrm{i}}, \tag{78}$$

$$C_0'' = a - \frac{b}{\omega\mathrm{i}} - \frac{F_0\Omega}{\omega(\Omega^2-\omega^2)\mathrm{i}}. \tag{79}$$

Solving (71), (72) one obtains

$$C_1' = D_1' + S', \qquad C_1'' = D_1'' + S'', \tag{80}$$

where

$$D_1' = \frac{\upsilon^2}{4\omega\mathrm{i}}\left[\frac{C_0'}{\upsilon}\,\mathrm{e}^{\mathrm{i}\upsilon T} + \frac{C_0'}{\upsilon}\,\mathrm{e}^{-\mathrm{i}\upsilon T} + \frac{C_0''}{\upsilon-2\omega}\,\mathrm{e}^{\mathrm{i}(\upsilon-2\omega)T} + \frac{C_0''}{\upsilon+2\omega}\,\mathrm{e}^{-\mathrm{i}(\upsilon+2\omega)T}\right]$$
$$+ \frac{F_0(\Omega^2-\omega^2+\upsilon^2)}{4\omega(\Omega^2-\omega^2)}\left[\frac{\mathrm{e}^{\mathrm{i}(\Omega-\omega+\upsilon)T}}{\Omega-\omega+\upsilon} - \frac{\mathrm{e}^{\mathrm{i}(\Omega-\omega-\upsilon)T}}{\Omega-\omega-\upsilon} + \frac{\mathrm{e}^{-\mathrm{i}(\Omega+\omega-\upsilon)T}}{\Omega+\omega-\upsilon} - \frac{\mathrm{e}^{-\mathrm{i}(\Omega+\omega+\upsilon)T}}{\Omega+\omega+\upsilon}\right], \tag{81}$$

$$D_1'' = -\frac{\upsilon^2}{4\omega\mathrm{i}}\left[\frac{C_0''}{\upsilon}\,\mathrm{e}^{\mathrm{i}\upsilon T} + \frac{C_0''}{\upsilon}\,\mathrm{e}^{-\mathrm{i}\upsilon T} + \frac{C_0'}{\upsilon-2\omega}\,\mathrm{e}^{-\mathrm{i}(\upsilon-2\omega)T} + \frac{C_0'}{\upsilon+2\omega}\,\mathrm{e}^{\mathrm{i}(\upsilon+2\omega)T}\right]$$
$$+ \frac{F_0(\Omega^2-\omega^2+\upsilon^2)}{4\omega(\Omega^2-\omega^2)}\left[\frac{\mathrm{e}^{-\mathrm{i}(\Omega-\omega+\upsilon)T}}{\Omega-\omega+\upsilon} - \frac{\mathrm{e}^{-\mathrm{i}(\Omega-\omega-\upsilon)T}}{\Omega-\omega-\upsilon} + \frac{\mathrm{e}^{\mathrm{i}(\Omega+\omega-\upsilon)T}}{\Omega+\omega-\upsilon} - \frac{\mathrm{e}^{\mathrm{i}(\Omega+\omega+\upsilon)T}}{\Omega+\omega+\upsilon}\right], \tag{82}$$

$$S' = -\frac{\upsilon^2}{2\omega\mathrm{i}}\left[\frac{C_0'}{\upsilon} + \frac{C_0''\upsilon}{\upsilon^2-4\omega^2}\right] + \frac{F_0\upsilon}{2\omega}\frac{\Omega^2-\omega^2+\upsilon^2}{\Omega^2-\omega^2}\left[\frac{1}{(\Omega-\omega)^2-\upsilon^2} - \frac{1}{(\Omega+\omega)^2-\upsilon^2}\right], \tag{83}$$

$$S'' = \frac{\upsilon^2}{2\omega\mathrm{i}}\left[\frac{C_0''}{\upsilon} + \frac{C_0'\upsilon}{\upsilon^2-4\omega^2}\right] + \frac{F_0\upsilon}{2\omega}\frac{\Omega^2-\omega^2+\upsilon^2}{\Omega^2-\omega^2}\left[\frac{1}{(\Omega-\omega)^2-\upsilon^2} - \frac{1}{(\Omega+\omega)^2-\upsilon^2}\right]. \tag{84}$$

The divisors in (81)–(84) indicate that in this system combination resonances can occur if $\Omega \pm \upsilon = \omega$ or $\upsilon - \Omega = \omega$. So, the proposed solutions are valid out of these combination resonances, when $\Omega \pm \upsilon \neq \omega$ and $\upsilon - \Omega \neq \omega$.

Analogously to the process of obtaining the adiabatic invariants for parametrically excited systems (26)–(28), but for the incomplete solution of the first component (73)–(75), the adiabatic invariants of the oscillator with combined parametric and forced excitation (68) are derived:

$$\left[X - \frac{\mathrm{i}P}{\omega}\right]\mathrm{e}^{-\mathrm{i}\omega T} - \frac{\mathrm{i}F_0}{2\omega}\left[\frac{\mathrm{e}^{\mathrm{i}(\Omega-\omega)T}}{\Omega-\omega} + \frac{\mathrm{e}^{-\mathrm{i}(\Omega+\omega)T}}{\Omega+\omega}\right] - \varepsilon D_1' = C_0' + \varepsilon S', \tag{85}$$

$$\left[X + \frac{\mathrm{i}P}{\omega}\right] \mathrm{e}^{\mathrm{i}\omega T} + \frac{\mathrm{i}F_0}{2\omega}\left[\frac{\mathrm{e}^{-\mathrm{i}(\Omega-\omega)T}}{\Omega-\omega} + \frac{\mathrm{e}^{\mathrm{i}(\Omega+\omega)T}}{\Omega+\omega}\right] - \varepsilon D_1'' = C_0'' + \varepsilon S''. \tag{86}$$

The complete forms of these invariants are the result of substituting (78)–(84) into (85) and (86). Since they are too cumbersome, they will not be shown here.

4.2. Resonant force

For the case when $\Omega = \omega$, the incomplete solution of (70) is given by (73), (74), but with

$$f(T,\tau) = C_0(\tau)\,\mathrm{e}^{B\omega^2 T} + \frac{BF_0}{4\omega}\,\mathrm{e}^{-B\omega^2 T} - \frac{F_0}{2\omega} T\,\mathrm{e}^{B\omega^2 T}. \tag{87}$$

The corresponding solution along trajectory is

$$U_0^* = \frac{C_0'(\tau)\,\mathrm{e}^{\mathrm{i}\omega T} + C_0''(\tau)\,\mathrm{e}^{-\mathrm{i}\omega T}}{2} + \frac{F_0}{4\omega^2}\frac{\mathrm{e}^{\mathrm{i}\omega T} - \mathrm{e}^{-\mathrm{i}\omega T}}{2\mathrm{i}} - \frac{F_0}{2\omega} T \frac{\mathrm{e}^{\mathrm{i}\omega T} + \mathrm{e}^{-\mathrm{i}\omega T}}{2}. \tag{88}$$

Using the assumed form for the second component of the field (77) and solving (71) we find that C_0' and C_0'' are constant:

$$C_0' = a + \frac{b}{\omega\mathrm{i}} + \frac{F_0}{4\omega\mathrm{i}}, \tag{89}$$

$$C_0'' = a - \frac{b}{\omega\mathrm{i}} - \frac{F_0}{4\omega\mathrm{i}}, \tag{90}$$

whereas

$$C_1' = K_1' + Q', \qquad C_1'' = K_1'' + Q'', \tag{91}$$

$$\begin{aligned} K_1' = {} & \frac{\upsilon^2}{4\omega\mathrm{i}}\left[\frac{C_0'}{\upsilon}\,\mathrm{e}^{\mathrm{i}\upsilon T} + \frac{C_0'}{\upsilon}\,\mathrm{e}^{-\mathrm{i}\upsilon T} + \frac{C_0''}{\upsilon-2\omega}\,\mathrm{e}^{\mathrm{i}(\upsilon-2\omega)T} + \frac{C_0''}{\upsilon+2\omega}\,\mathrm{e}^{-\mathrm{i}(\upsilon+2\omega)T}\right] \\ & - \frac{F_0(\upsilon^2-4\omega^2)}{16\omega^3}\left[\frac{\mathrm{e}^{\mathrm{i}\upsilon T}}{\upsilon} + \frac{\mathrm{e}^{-\mathrm{i}\upsilon T}}{\upsilon} - \frac{\mathrm{e}^{\mathrm{i}(\upsilon-2\omega)T}}{\upsilon-2\omega} - \frac{\mathrm{e}^{-\mathrm{i}(\upsilon+2\omega)T}}{\upsilon+2\omega}\right] \\ & - \frac{F_0\upsilon^2}{8\omega^2}\left[\frac{T}{\mathrm{i}}\left(\frac{\mathrm{e}^{\mathrm{i}\upsilon T}}{\upsilon} + \frac{\mathrm{e}^{-\mathrm{i}\upsilon T}}{\upsilon} + \frac{\mathrm{e}^{\mathrm{i}(\upsilon-2\omega)T}}{\upsilon-2\omega} + \frac{\mathrm{e}^{-\mathrm{i}(\upsilon+2\omega)T}}{\upsilon+2\omega}\right) + \frac{\mathrm{e}^{\mathrm{i}\upsilon T}}{\upsilon^2} - \frac{\mathrm{e}^{-\mathrm{i}\upsilon T}}{\upsilon^2}\right. \\ & \left. + \frac{\mathrm{e}^{\mathrm{i}(\upsilon-2\omega)T}}{(\upsilon-2\omega)^2} - \frac{\mathrm{e}^{-\mathrm{i}(\upsilon+2\omega)T}}{(\upsilon+2\omega)^2}\right], \end{aligned} \tag{92}$$

$$\begin{aligned} K_1'' = {} & -\frac{\upsilon^2}{4\omega\mathrm{i}}\left[\frac{C_0''}{\upsilon}\,\mathrm{e}^{\mathrm{i}\upsilon T} + \frac{C''}{\upsilon}\,\mathrm{e}^{-\mathrm{i}\upsilon T} + \frac{C_0'}{\upsilon-2\omega}\,\mathrm{e}^{-\mathrm{i}(\upsilon-2\omega)T} + \frac{C_0'}{\upsilon+2\omega}\,\mathrm{e}^{\mathrm{i}(\upsilon+2\omega)T}\right] \\ & - \frac{F_0(\upsilon^2-4\omega^2)}{16\omega^3}\left[\frac{\mathrm{e}^{\mathrm{i}\upsilon T}}{\upsilon} + \frac{\mathrm{e}^{-\mathrm{i}\upsilon T}}{\upsilon} - \frac{\mathrm{e}^{-\mathrm{i}(\upsilon-2\omega)T}}{\upsilon-2\omega} - \frac{\mathrm{e}^{\mathrm{i}(\upsilon+2\omega)T}}{\upsilon+2\omega}\right] \\ & + \frac{F_0\upsilon^2}{8\omega^2}\left[\frac{T}{\mathrm{i}}\left(-\frac{\mathrm{e}^{\mathrm{i}\upsilon T}}{\upsilon} - \frac{\mathrm{e}^{-\mathrm{i}\upsilon T}}{\upsilon} + \frac{\mathrm{e}^{-\mathrm{i}(\upsilon-2\omega)T}}{\upsilon-2\omega} + \frac{\mathrm{e}^{\mathrm{i}(\upsilon+2\omega)T}}{\upsilon+2\omega}\right) + \frac{\mathrm{e}^{\mathrm{i}\upsilon T}}{\upsilon^2} - \frac{\mathrm{e}^{-\mathrm{i}\upsilon T}}{\upsilon^2}\right. \\ & \left. - \frac{\mathrm{e}^{-\mathrm{i}(\upsilon-2\omega)T}}{(\upsilon-2\omega)^2} + \frac{\mathrm{e}^{\mathrm{i}(\upsilon+2\omega)T}}{(\upsilon+2\omega)^2}\right], \end{aligned} \tag{93}$$

$$Q' = -\frac{\upsilon^2}{2\omega\mathrm{i}}\left[\frac{C_0'}{\upsilon} + \frac{C_0''\upsilon}{\upsilon^2-4\omega^2}\right] + \frac{F_0}{2\omega\upsilon} + \frac{F_0\upsilon^3}{\omega(\upsilon-2\omega)^2(\upsilon+2\omega)^2}, \tag{94}$$

$$Q'' = \frac{\upsilon^2}{2\omega\mathrm{i}}\left[\frac{C_0''}{\upsilon} + \frac{C_0'\upsilon}{\upsilon^2-4\omega^2}\right] - \frac{F_0}{2\omega\upsilon} + \frac{F_0\upsilon^3}{\omega(\upsilon-2\omega)^2(\upsilon+2\omega)^2}. \tag{95}$$

Finally, transforming the field $X \equiv U = U_0 + \varepsilon U_1$ by using the incomplete solution for the first component (73), (74), (87) and the incomplete solution for the second one (77), it follows

$$\left[X - \frac{\mathrm{i}P}{\omega}\right]\mathrm{e}^{-\mathrm{i}\omega T} - \frac{\mathrm{i}F_0}{4\omega^2}\,\mathrm{e}^{-2i\omega T} + \frac{F_0}{2\omega}T - \varepsilon K_1' = C_0' + \varepsilon Q', \tag{96}$$

$$\left[X + \frac{\mathrm{i}P}{\omega}\right]\mathrm{e}^{\mathrm{i}\omega T} + \frac{\mathrm{i}F_0}{4\omega^2}\,\mathrm{e}^{2i\omega T} - \frac{F_0}{2\omega}T - \varepsilon K_1'' = C_0'' + \varepsilon Q''. \tag{97}$$

5. Conclusion

In this study, a novel approach to deriving the complete set of adiabatic invariants for some time-dependent 1D oscillators is presented. It is based on the field method concept of deriving an invariant from an incomplete solution of a partial differential equation. This concept is adopted by combining it with the multiple variable expansion procedure. It has been shown how the linear adiabatic invariants and the corresponding quadratic invariant can be derived for the parametrically excited oscillator and the parametrically pumped oscillator excited with a periodic driving force.

The procedure proposed for the parametrically excited oscillator whose motion is described by the Mathieu equation enables obtaining the complete set of linear adiabatic invariants and, consequently, the approximate solution for motion. Moreover, it gives the expressions for the transition curves and makes it possible to carry out the stability analysis (see the paragraph after equation (31)). All these benefits make the field method approach more fruitful in comparison to the method of strained parameters and the method of multiple scales. Namely, the method of strained parameters gives the expressions for the transition curves and the approximate solution for motion, while the method of multiple scales enables finding the transition curves, the approximate solution for motion and carrying out the stability analysis.

The parametrically pumped oscillator described by the Kanai–Caldirola Hamiltonian is treated with the aim of obtaining its adiabatic invariants in the case when the non-resonant force acts as well as in the case when the forced resonance occurs. Besides, the possibilities of combination resonance are detected and the combinations of the system parameters which lead to this phenomenon are obtained.

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